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On the strong law of large numbers for Ornstein-Uhlenbeck type processes with increasing perturbations and its application to a stochastic control problem

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We establish the strong law of large numbers (SLLN) for an integrated squared Ornstein-Uhlenbeck type process with time-varying coefficients and asymptotically unbounded diffusion matrix. The result is applied to study an extention of the ergodic control to linear stochastic systems with increasing perturbations.

We start with the basic definition of the SLLN for a pair of stochastic processes (see [3]).

Definition 1. Let $\{Y_t\}_{t=0}^{\infty}$ and $\{L_t\}_{t=0}^{\infty}$ with $L_0 = 0$ be a semimartingale and a predictable increasing process defined on a stochastic basis $\{\Omega, \mathcal{F}, (F_t)_{t\ge 0}, \mathbf{P}\}$. We say that the pair (Y_t, L_t) satisfies the *SLLN* if $P(\lim\limits_{t\to\infty}\{Y_t/L_t\} = 0) = 1$.

Consider an n-dimensional stochastic process X_t, $t \ge 0$, given by

$$dX_t = A_t X_t\, dt + G_t\, dw_t\,, \quad X_0 = x\,, \tag{1}$$

where $A_t \in R^{n\times n}$, $G_t \in R^{n\times d}$ are non-random matrix functions, A_t is bounded but $\|G_t\| \to \infty$, $t \to \infty$; w_t, $t \ge 0$, is a d-dimensional standard Wiener process; x is a non-random vector.

First let us make the following assumption.

Assumption $\mathcal{A}$. The matrix A_t is exponentially stable, i.e. there exist positive constants κ_1, κ_2 such that $\|\Phi(t,s)\| \le \kappa_1 e^{-\kappa_2(t-s)}$, for all $s \le t$, where $\Phi(t,s)$ is the fundamental matrix corresponding to A_t, $\|\cdot\|$ denotes the Euclidean matrix norm.

Note that under the Assumption $\mathcal{A}$ and with $\|G_t\| \to \infty$ as $t \to \infty$, (1) can be used in the modeling of anomalous diffusion processes [4]. The conditions imposed on G_t are given below.

Assumption $\mathcal{G}$. The diffusion matrix G_t is such that

a) $\|G_t\|$ is increasing, $\|G_t\| \to \infty$, $t \to \infty$;

b) $\ln\|G_t\|$ is strictly concave;

c) $\lim\limits_{t\to\infty}(\|G_t\|^2 \ln t / \int\limits_0^t \|G_s\|^2\, ds) = 0$.

The main result is established in the next Theorem.

Theorem 1. Let $L_t = \int\limits_0^t \|G_s\|^2\, ds$, $Y_t = \int\limits_0^t (\|X_s\|^2 - E\|X_s\|^2)\, ds$, where X_t is a solution to (1). Assuming $\mathcal{A}$ and $\mathcal{G}$, the pair (Y_t, L_t) satisfies the *SLLN*.

Remark. Take L_t defined in Theorem 1. Under $\mathcal{A}$ and **a)**,**c)** from $\mathcal{G}$, the SSLN holds for (Y_t, L_t) with $Y_t = \|X_t\|^2$. If $\mathcal{A}$ and $\mathcal{G}$ **b)** are satisfied, then we have the SSLN for $Y_t = \int\limits_0^t X_s' G_s dw_s$, where $'$ denotes the transpose.

Next we turn to the study of a pathwise optimal control problem for linear stochastic systems. The controlled stochastic process $Z_t, t \ge 0$, is governed by

$$dZ_t = C_t X_t dt + B_t U_t dt + G_t dw_t\,, \qquad Z_0 = z, \tag{2}$$

where C_t, B_t are bounded, G_t satisfies $\mathcal{G}$; U_t, $t \ge 0$, is an admissible control, i.e. an $\mathcal{F}_t = \sigma\{w_s,\, s \le t\}$–adapted k-dimensional process such that there exists a solution to (2). Let us denote by $\mathcal{U}$ the set of admissible controls. The cost functional over the planning horizon $[0,T]$ is given by

$$J_T(U) = \int_0^T [Z_t' Q_t Z_t + U_t' R_t U_t]\, dt, \tag{3}$$

where $U \in \mathcal{U}$; Q_t, R_t are symmetric matrix functions, Q_t is positive semidefinite and R_t is positive definite.

Definition 2. A control $U^* \in \mathcal{U}$ is called *pathwise optimal over an infinite time horizon* if it is a solution to

$$\limsup_{T\to\infty} \frac{J_T(U)}{\int\limits_0^T \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}} \text{ almost surely}\,. \tag{4}$$

If we took $G_t = G$, the criterion in (4) would be equal to the pathwise ergodic (pathwise average), see [2], so our problem extends the notion of ergodic control to linear stochastic systems with unbounded perturbation parameters.

We will use the conditions given in [1] (see Assumptions **1**, **2** there) that, as can be proved, guarantee the existence of the optimal control U^*. Assumption 1 from [1] states that there exists the bounded absolute continuous function Π_t, $t \geq 0$ with values in the set of symmetric positive semidefinite matrices, which satisfies the Riccati equation

$$\dot{\Pi}_t + \Pi_t C_t + C_t' \Pi_t - \Pi_t B_t R_t^{-1} B_t' \Pi_t + Q_t = 0\,, \tag{5}$$

and the matrix $C_t - B_t R_t^{-1} B_t'$ is exponentially stable. Assuming **1**, **2** from [1], define the feedback control law by

$$U_t^* = -R_t^{-1} B_t' \Pi_t Z_t^*\,, \tag{6}$$

where the process $Z_t^*, t \geq 0$, is given by

$$dZ_t^* = (C_t - B_t R_t^{-1} B_t' \Pi_t) Z_t^* dt + G dw_t\,,\ Z_0^* = z. \tag{7}$$

We proved the following result.

Theorem 2. Suppose the Assumptions $\mathcal{G}$ and **1**, **2** from [1] hold. If $\lim\limits_{t\to\infty} (\|G_t\|^2 \ln^2 \ln \|G_t\|) / \int\limits_0^t \|G_s\|^2\, ds) = 0$, then

a) the control U^* defined by (6)–(7) is pathwise optimal over an infinite time horizon

b) $\limsup\limits_{T\to\infty} \dfrac{J_T(U^*)}{\int\limits_0^T \|G_t\|^2\, dt} = \limsup\limits_{T\to\infty} \dfrac{E J_T(U^*)}{\int\limits_0^T \|G_t\|^2\, dt}$ is finite.

Источники и литература

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